Clique Immersion in Digraphs

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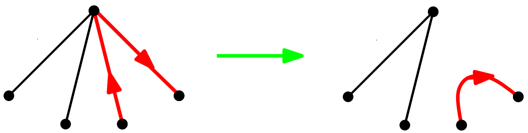
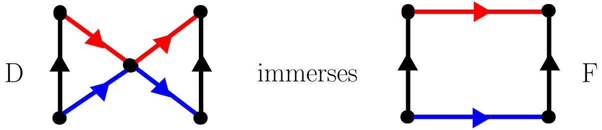
What is digraph immersion?

Given two digraphs D, F, we say "D has an F-immersion" if:

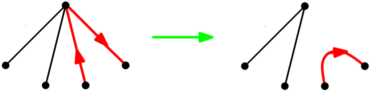
F can be obtained from D via deletion and splitting-off edges.



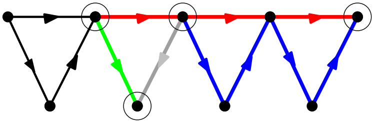
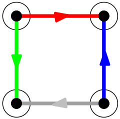
Equivalently: "D immerses F" or "F is immersed in D" or "D contains F as an immersion"



Equivalent definition:

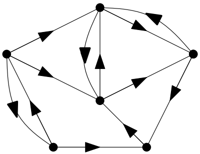


D has an F-immersion if
there exists an injective map $\phi : V(F) \to V(D)$ and a collection of edge-disjoint directed paths in D, one from $\phi(u)$ to $\phi(v)$ for every edge uv in F.

 contains  as an immersion.

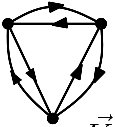
- The vertices $\{\phi(v) : v \in V(F)\}$ are called *terminals*.

- The collection of directed paths may not be internally disjoint from the set of terminals. If they are, we have a *strong* immersion.



Today we ask:

When does a digraph D have a $\vec{K}_t$-immersion?



$\vec{K}_3$

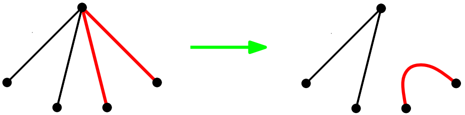
Outline

1. Moving from graphs to digraphs
2. Two proofs of one theorem
3. Two theorems, one corollary
4. Conclusion

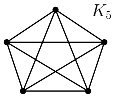
1 Moving from graphs to digraphs

Given two *graphs* G, H, we say "G has an H-immersion" if:

H can be obtained from G via deletion and splitting-off edges.



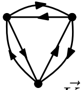
Theorem (DeVos, Dvořák, Fox, M., Mohar, Scheide)
Every simple graph with minimum degree $\geq$ 200t has a (strong) K_t-immersion.



- 200t can be lowered to t-1 (best possible) when $t \leq 7$ (Lescure and Mayniel; DeVos, Kawarabayashi, Mohar, Okamura)
- t-1 does not suffice when $t \geq 10$ (Seymour)

Immersion is harder is digraphs...but is there an analogous result?

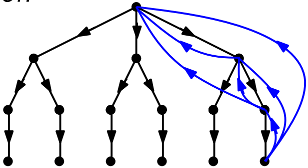
Theorem (DeVos, M., Mohar, Scheide)
For every $k \in \mathbb{Z}^+$, there exists a digraph with minimum in-degree and out-degree $\geq k$, but no $\overrightarrow{K_3}$-immersion.



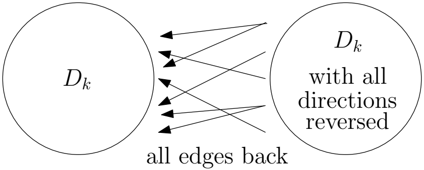
$\vec{K}_3$

proof:

D_3



Add an arc from each vertex to all those vetices between it and the root (and to the root)

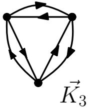


What went wrong?

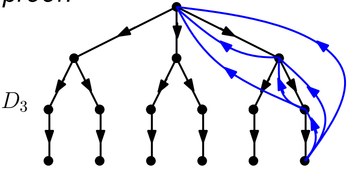
- To split an vertex completely: Eulerian.
- Graphs: $\delta \geq 2k \supseteq$ Eulerian, $\delta \geq k$ (Tutte, Nash-Williams).
- The above example is very far from being Eulerian.

Theorem (DeVos, M., Mohar, Scheide)

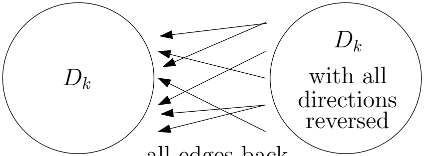
For every $k \in \mathbb{Z}^+$, there exists a digraph with minimum in-degree and out-degree $\geq k$, but no $\overrightarrow{K_3}$-immersion.



proof:



Add an arc from each vertex to all those vetices between it and the root (and to the root)



all edges back

What went wrong?

- To split an vertex completely: Eulerian.
- Graphs: $\delta \geq 2k \supseteq$ Eulerian, $\delta \geq k$ (Tutte, Nash-Williams).
- The above example is very far from being Eulerian.

Theorem (DeVos, Dvořák, Fox, M., Mohar, Scheide).
Every simple graph with minimum degree at least 200t has a strong K_t-immersion.

Theorem (DeVos, M., Mohar, Scheide)
For every $k \in \mathbb{Z}^+$, there exists a digraph with minimum in-degree and out-degree $\geq k$, but no $\overrightarrow{K_3}$-immersion.

Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph with minimum degree at least t(t-1) contains a $\vec{K}_t$-immersion.

Theorem (DeVos, M., Mohar, Scheide).
If $t \leq 4$, every simple Eulerian digraph with minimum degree at least $t-1$ contains a $\vec{K}_t$-immersion (and this is best possible).

2 Two proofs of one theorem

Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph with minimum degree at least t(t-1) contains a $\vec{K}_t$-immersion.

Proof Ingredients	Proof 1	Proof 2
Edmonds' Disjoint Arborescence Theorem	✓	✓
Mader's Directed Splitting Theorem	✓	✓
The Gomery-Hu Theorem	X	✓

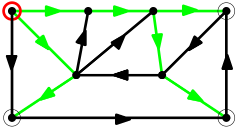
Arborscense w. root v = spanning tree, all edges away from v

Edmonds' Disjoint Aborescence Theorem.
Let $v_1, \ldots v_r$ be vertices in a digraph D (not necessarily distinct). Then $\exists$ edge-disjoint arborescences $T_1, \ldots T_r$ so that T_i has root v_i iff every $X \subset V(D)$ satisfies $d^+(X) \geq |\{i : v_i \in X, 1 \leq i \leq r\}|$.

Corollary. If a digraph is strongly $t(t-1)$-edge connected with at least t vertices then it contains a $\overrightarrow{K_t}$-immersion.

proof:

- Let $v_1, \ldots, v_t$ be distinct, then choose each $t-1$ times.
- We may apply Edmonds' Thm to this set of $t(t-1)$ vertices.
- We get $t(t-1)$ edge-disjoint arborescences, $t-1$ of which are rooted at v_i.
- These $t-1$ arborescences give edge-disjoint paths from v_i.



Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph D with minimum degree at least t(t-1) contains a $\vec{K}_t$-immersion.

Proof Ingredients	Proof 1	Proof 2
strongly t(t-1)-edge-connected, $\geq t$ vertices $\Rightarrow$ $\vec{K}_t$-imm.	✓	✓
Mader's Directed Splitting Theorem	✓	✓
The Gomery-Hu Theorem	X	✓

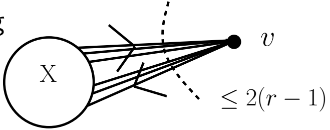
Proofs 1 & 2:

It suffices to show that D immerses a strongly $t(t-1)$-edge-connected digraph on a least t vertices.

Proof 1 (sketch): Let $t(t-1) = r$. D simple, Eul., min. deg. $\geq r$. Show D immerses strongly r-edge-connected digraph, $\geq t$ vertices.

• We are able to find $X \subseteq D$ such that all pairs of vertices in X are sufficiently connected (but perhaps through all of D), and...

• ...we are able to immerse the following Eulerian digraph in D (maintaining connectivity between pairs):



Mader's Directed Splitting Theorem.
Given an Eulerian digraph and a non-isolated vertex w, there is a pair of edges that can be split off of w so the size of the smallest edge-cut between any other pair of vertices doesn't change.

• Use Mader's Theorem to split v completely.

• At most $r-1$ parallel edges. Coupled with the min. deg. condition, this says we have $r \geq t$ vertices. $\square$

Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph D with minimum degree at least t(t-1) contains a $\vec{K}_t$-immersion.

Proof Ingredients	Proof 1	Proof 2
strongly t(t-1)-edge-connected, $\geq t$ vertices $\Rightarrow$ $\vec{K}_t$-imm.	✓	✓
Mader's Directed Splitting Theorem	✓	✓
The Gomery-Hu Theorem	X	✓
	✓	

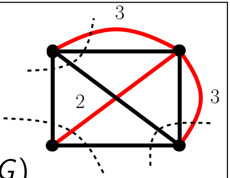
Proofs 1 & 2:

It suffices to show that D immerses a strongly $t(t-1)$-edge-connected digraph on a least t vertices.

Proof 2: D simple, Eulerian, min. deg. $\geq t(t-1)$. Show D immerses a strongly $t(t-1)$-edge-connected digraph on $\geq t$ vertices.

The Gomory-Hu Theorem.
For every multigraph G there exists a tree F with vertex set $V(G)$ and a function $\mu : E(F) \to \mathbb{Z}$ such that:

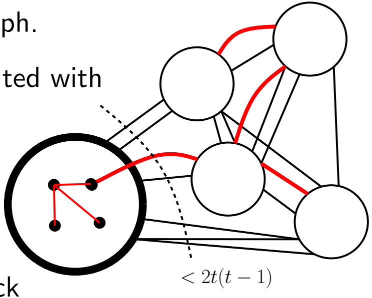
- $\lambda_G(u, v) = \min\{\mu(e) : e \in uFv\} \quad \forall u, v \in V(G)$
- $\mu(e) = |\text{edge cut of } G \text{ associated with e}| \quad \forall e \in E(F)$



Apply GH to the underlying multigraph.

The family of edge-cuts of D associated with $\{e \in E(F) : \mu(e) < 2t(t-1)\}$ induces a partition of $V(D)$.

Blocks of the partition must have size $\geq t$ by simple, min. degree.



Choose t distinct vertices in one block and apply Mader's Theorem to split all other vertices completely. $\square$

Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph D with minimum degree at least t(t-1) contains a $\vec{K}_t$-immersion.

Proof Ingredients	Proof 1	Proof 2
strongly t(t-1)-edge-connected, $\geq t$ vertices $\Rightarrow$ $\vec{K}_t$-imm.	✓	✓
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The Gomery-Hu Theorem	X	✓
	✓	✓

Proofs 1 & 2:

It suffices to show that D immerses a strongly $t(t-1)$-edge-connected digraph on a least t vertices...

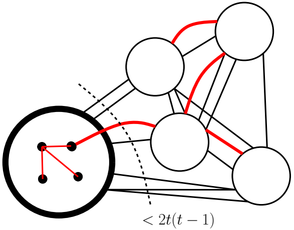
...but in both cases we proved something better.

3 Two Theorems, One Corollary

Theorem (DeVos, Mohar, M., Scheide).
Every simple Eulerian digraph with minimum degree $\geq r$ immerses a strongly r-edge-connected digraph on at least r vertices.

Theorem (DeVos, Mohar, M., Scheide).
If D is an Eulerian digraph with no $\vec{K}_t$-immersion then it has a laminar family of edge cuts, each with size $< 2t(t-1)$, so that every block of the resulting partition has size less than t.

Corollary.
Every simple Eulerian digraph D with minimum degree at least t(t-1) contains a $\vec{K}_t$-immersion.



Theorem (DeVos, Mohar, M., Scheide).
If D is an Eulerian digraph with no $\vec{K}_t$-immersion then it has a laminar family of edge cuts, each with size $< 2t(t-1)$, so that every block of the resulting partition has size less than t.

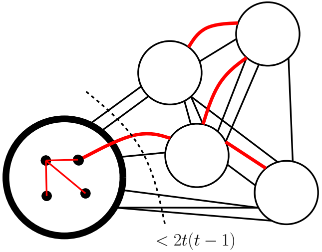
$\uparrow$

Rough structure for $\vec{K}_t$-immersion.

(Backwards: no $\vec{K}_{t^2}$, K_{t^2})

Rough structure for K_t-immersion.

$\downarrow$



Theorem (Seymour, Wollan).
If G is graph with no K_t-immersion then it has a laminar family of edge cuts, each with size $< (t-1)^2$, so that every block of the resulting partition has size less than t.

4 Conclusion

Immersing a clique is harder in digraphs than it is in graphs:

- We need to take Eulerian as an assumption.
- Can we lower min. deg. $\geq t(t-1)$ to linear?

Along with the minimum degree result, we get:

- Simple Eulerian digraph with min. deg. $\geq r$ immerses a strongly r-edge-connected digraph with $\geq r$ vertices.
- Rough structure theorem for $\vec{K}_t$-immersion.

Thank-you

More questions?

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